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THE METHOD OF AVERAGING IN A CONTACT PROBLEM FOR A SYSTEM OF PUNCHES†

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The method of matched asymptotic expansions is used to study a contact problem for a system consisting of a large number of small punches situated along a given curve on the boundary of an elastic half-space. The cases of cylindrical punches (the linear problem) and spherical punches (the structurally non-linear contact problem) are considered. In the linear case the reduced logarithmic capacity of the contact area is shown to possess the property of monotonicity and its asymptotic behaviour is determined. The resultant integral equations for the average density of linear pressures are derived.

1. FORMULATION OF THE LINEAR CONTACT PROBLEM

Let Γ be a simple smooth closed curve of length $2l$ in the (x_1, x_2) plane. Let (s, n) be a local system of coordinates introduced in its neighbourhood, where s is the length of the arc and n is the distance (taking the sign into account) along the interior normal. Let N be a large natural number and let $\varepsilon = 1/N$ be a small parameter. Let ω_1 denote a domain in the plane of "expanded" coordinates (ξ_1, ξ_2) contained in a disk of radius l, and let ω_2 be the domain obtained from ω_1 by N-fold contraction. We define a periodically varying narrow set along the contour Γ by

$$\Gamma(\varepsilon) = \left\{(x_1, x_2): \left(s - j\frac{2l}{N}, n\right) \in \omega_e, j = 0, 1, \ldots, N-1\right\} \tag{1.1}$$

In other words, $\Gamma(\varepsilon)$ is the union of N pairwise disjoint domains ω_ε^j $(j = 0, 1, \ldots, N-1)$ of small diameter (of order of magnitude $2\varepsilon l$) that have the form of the domain ω_ε in local coordinates.

The contact problem of a punch with flat base $\Gamma(\varepsilon)$ pressed without friction into an elastic half-space $x_3 \le 0$ (with Young's modulus E and Poisson's ratio ν) reduces via the Papkovich–Neuber representation [1] to the problem

$$\Delta_x\varphi(\mathbf{x}) = 0, \quad x_3 < 0; \quad \partial_3\varphi(\mathbf{x}) = 0, \quad x_3 = 0 \quad (x_1, x_2) \notin \overline{\Gamma(\varepsilon)} \tag{1.2}$$

$$\varphi(x_1, x_2, 0) = -\delta_0 - \beta_2 x_1 + \beta_1 x_2 \quad (x_1, x_2) \in \Gamma(\varepsilon) \tag{1.3}$$

$$\varphi(\mathbf{x}) = o(1), \quad |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty \tag{1.4}$$

where δ_0 and β_1, β_2 are the translational displacement and angles of rotation of the punch $\Gamma(\varepsilon)$ relative to the horizontal coordinate axes, and $\partial_3 = \partial/\partial x_3$.

The pressure exerted on a semi-infinite elastic body by the punch is computed from the formula

$$p(x_1, x_2) = -E[2(1-\nu^2)]^{-1}\partial_3\varphi(x_1, x_2, 0) \quad (x_1, x_2) \in \Gamma(\varepsilon) \tag{1.5}$$

An asymptotic analysis of the contact problem has been constructed for a system of punches densely situated within a bounded area on the surface of an elastic half-space [2]. In this paper the singularly perturbed contact problem (1.2)–(1.4), corresponding to the case of a chain of punches situated along a given curve, is averaged by a method developed by Nazarov [3]. The distinctive feature of this problem is the need to use the method of matched asymptotic expansions (see [4–6], etc.) instead of the method

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of composite asymptotic expansions [7], which was used previously [3] when deriving the problem for a boundary layer.

2. CONSTRUCTION OF AN ASYMPTOTIC FORM IN THE LINEAR CONTACT PROBLEM

At a distance from the contact area $\Gamma(\varepsilon)$, the function $\varphi(\mathbf{x})$ may represented for the most part as a simple layer potential, whose density is spread over the contour Γ,

$$v(\gamma; \mathbf{x}) = -\frac{1}{2\pi}\int_{\Gamma}\frac{\gamma(t)dt}{\sqrt{(x_1 - f_1(t))^2 + (x_2 - f_2(t))^2 + x_3^2}} \tag{2.1}$$

where dt is the element of arc length and the equations $x_1 = f_1(t)$, $x_2 = f_2(t)$ define a natural parameterization of the contour Γ. (To fix our ideas, we shall assume that when Γ is described in the sense of increasing coordinate s, the domain bounded by the contour Γ remains on the left). The function $\gamma(s)$ has to be defined. The quantity $E[2(1-\nu^2)]^{-1}\gamma(s) \equiv P(s)$ has the meaning of the average linear contact pressure.

In planes normal to Γ we introduce polar coordinates (r, φ) so that $n = r\cos\varphi$, $x_3 = r\sin\varphi$, $\varphi \in [-\pi, 0]$. On the assumption that the function γ is continuously differentiable, the following formula holds as $r \to 0$ (see, e.g., [8])

$$v(\gamma; s, r, \varphi) = \frac{\gamma(s)}{\pi}\left(\ln\frac{r}{2l} - J^0(s)\right) - \frac{1}{\pi}(J\gamma)(s) + O(r\ln(k_m r)) \tag{2.2}$$

where k_m is the maximum curvature of the contour Γ, and we have also introduced the notation

$$J^0(s) = \frac{1}{2}\int_{s-l}^{s+l}(R_0(s, t)^{-1} - |s-t|^{-1})dt \tag{2.3}$$

$$(J\gamma)(s) = \frac{1}{2}\int_{\Gamma}\frac{\gamma(t)-\gamma(s)}{R_0(s, t)}dt$$
$$R_0(s, t) = [(f_1(s) - f_1(t))^2 + (f_2(s) - f_2(t))^2]^{1/2} \tag{2.4}$$

The function of two variables $R_0(s, t)$ defines the difference between two points on Γ with coordinates s and t, and moreover

$$R_0(s, t) = |s-t|(1 + O(k_m^2|s-t|^2)), \quad t \to s \tag{2.5}$$

In the neighbourhood of the contact area $\Gamma(\varepsilon)$ one has a boundary-layer effect, that is, the solution $\varphi(\mathbf{x})$ of problem (1.2)–(1.4) shows characteristic sharp variations in a small neighbourhood of the narrow set $\Gamma(\varepsilon)$. A function $w(s; \xi^j)$ of the boundary-layer type describing this behaviour of the solution of the initial problem depends both on the "slow" variable s and on expanded coordinates

$$\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j), \quad \xi^j = \varepsilon^{-1}(s - s_\varepsilon^j, n, x_3); \quad s_\varepsilon^j = j2lN^{-1} \tag{2.6}$$

Following the algorithm described previously in [3], we arrive at the equation

$$\Delta_\xi w(s_\varepsilon^j; \xi) = 0, \quad \xi_3 < 0; \quad |\xi_1| < l \tag{2.7}$$

in a half-layer of thickness $2l$. From now on, the superscript j in the symbol ξ^j will not be written, since there is no difference between the equations comprising the problem for a boundary layer for different values of j. Moreover, the discrete variable $s_\varepsilon^j \in \{j2\varepsilon l, j = 0, 1, \dots, N-1\}$ will be replaced by a continuous parameter $s \in [0, 2l)$.

According to Eqs (1.2) and (1.3), the function $w(s; \xi)$ must satisfy the boundary conditions

$$\frac{\partial w}{\partial \xi_3}(s; \xi) = 0, \quad \xi_3 = 0, \ (\xi_1, \xi_2) \notin \bar{\omega}_1, \quad |\xi_1| < l \tag{2.8}$$

$$w(s;\xi_1,\xi_2,0) = w_0(s), \quad (\xi_1,\xi_2)\in\omega_1 \tag{2.9}$$

where we have introduced the notation

$$w_0(s) = -\delta_0 - \beta_2 f_1(s) + \beta_1 f_2(s) \tag{2.10}$$

In addition, the following compatibility conditions must hold

$$w(s;-l,\xi_2,\xi_3) = w(s;l,\xi_2,\xi_3), \quad \frac{\partial w}{\partial \xi_1}(s;-l,\xi_2,\xi_3) = \frac{\partial w}{\partial \xi_1}(s;l,\xi_2,\xi_3), \quad \xi_3<0 \tag{2.11}$$

Finally, in view of the asymptotic formula (2.2), the matching condition for the inner asymptotic representation $w(s;\xi)$ and the outer one $\upsilon(\gamma;\mathbf{x})$ defined by formula (2.1) leads to the following asymptotic condition at infinity

$$w(s;\xi) = \frac{\gamma(s)}{\pi}\left(\ln\frac{\varepsilon\rho}{2l} - J^0(s)\right) - \frac{1}{\pi}(J\gamma)(s) + O(\rho^{-1}), \quad \rho\to\infty \tag{2.12}$$

where we have introduced the notation $\rho = \sqrt{\xi_1^2+\xi_2^2}$, with $\rho = \varepsilon^{-1}r$, r being the distance to the contour Γ. Note also that the coordinate s occurs in (2.9) and (2.12) as a parameter.

Given the solution $w(s;\xi)$ of problem (2.8)–(2.12), the contact pressure under the punch base ω_ε^j in expanded coordinates is calculated using the formula (see formulae (1.5) and (2.6))

$$p(x_1,x_2) \simeq -\frac{E}{2(1-\nu^2)}\frac{1}{\varepsilon}\frac{\partial w}{\partial \xi_3^j}(s;\xi_1^j,\xi_2^j,0) \quad (\xi_1^j,\xi_2^j)\in\omega_1 \tag{2.13}$$

3. THE LOGARITHMIC CAPACITY OF THE CONTACT AREA

An existence and uniqueness theorem has been proved [3] for the solution $e(\xi)$ of the homogeneous problem (2.7)–(2.10), which goes to infinity as $-\ln\rho$. The constant in the asymptotic formula

$$e(\xi) = -\ln\rho + \kappa_1 + O(\rho^{-1}), \quad \rho\to\infty \tag{3.1}$$

is related to the so-called reduced logarithmic capacity $c_{\log}(\omega_1)$ of the set $\{\xi:\xi_3=0, (\xi_1,\xi_2)\in\bar{\omega}_1\}$ [3] by the formula

$$c_{\log}(\omega_1) = \exp(\kappa_1) \tag{3.2}$$

According to the maximum principle for harmonic functions (see, e.g., [9]), the following relations hold

$$\frac{\partial e}{\partial \xi_3}(\xi_1,\xi_2,0-)>0, \quad (\xi_1,\xi_2)\in\omega_1 \tag{3.3}$$

$$e(\xi_1,\xi_2,0)<0 \quad (\xi_1,\xi_2)\notin\bar{\omega}_1, \quad |\xi_1|\le l \tag{3.4}$$

In addition, by applying Green's formula it can be shown that

$$\frac{1}{2\pi l}\iint_{\omega_1}\frac{\partial e}{\partial \xi_3}(\xi_1,\xi_2,0-)d\xi_1 d\xi_2 = 1 \tag{3.5}$$

It has been shown [3] that a unique bounded solution of problem (2.7), (2.8), (2.11) exists that satisfies the inhomogeneous boundary condition

$$w(s;\xi_1,\xi_2,0) = w_0(s;\xi_1,\xi_2) \quad (\xi_1,\xi_2)\in\omega_1$$

and admits of an asymptotic expansion

$$w(s;\xi) = c^0(s) + O(\rho^{-1}), \quad \rho\to\infty \tag{3.6}$$

Under these conditions the quantity $c^0(s)$ in formula (3.6) may be expressed, using the Maz'ya–Plamenevskii method [10], as follows:

$$c^0(s) = \frac{1}{2\pi l}\iint\limits_{\omega_1} w_0(s;\xi_1,\xi_2)\frac{\partial e}{\partial \xi_3}(\xi_1,\xi_2,0-)d\xi_1 d\xi_2 \quad (3.7)$$

Let us consider two domains, ω_1' and ω_2'', in the strip $|\xi_1| < l$, one of which contains the other, that is, $\omega_1' \subset \omega_1''$, where the area of the set $\omega_1''\backslash\overline{\omega_1'}$ is positive. Their reduced logarithmic capacities satisfy the inequality

$$c_{\log}(\omega_1') < c_{\log}(\omega_1'') \quad (3.8)$$

Indeed, consider the solution $e'(\xi)$ and $e''(\xi)$ of the problems (2.7)–(2.11) which increase at infinity as $-\ln\rho$ and satisfy Dirichlet boundary conditions (2.9) given on the domains ω_1' and ω_1''. The difference $w(\xi) = e''(\xi) - e'(\xi)$ satisfies the Laplace condition (2.7), the periodicity conditions (2.11), and the following relations

$$w(\xi_1,\xi_2,0) = 0, \quad (\xi_1,\xi_2)\in\omega_1'$$

$$w(\xi_1,\xi_2,0) = -e'(\xi_1,\xi_2,0), \quad (\xi_1,\xi_2)\in\omega_1''\backslash\overline{\omega_1'}$$

$$\frac{\partial w}{\partial \xi_3}(\xi_1,\xi_2,0) = 0, \quad (\xi_1,\xi_2)\notin\overline{\omega_1''}, \quad |\xi_1|<l$$

$$w(\xi) = \kappa_1'' - \kappa_1' + O(\rho^{-1}), \quad \rho\to\infty$$

By formula (3.7), we have

$$\kappa_1'' - \kappa_1' = -\frac{1}{2\pi l}\iint\limits_{\omega_1''/\overline{\omega_1'}} e'(\xi_1,\xi_2,0)\frac{\partial e''}{\partial \xi_3}(\xi_1,\xi_2,0-)d\xi_1 d\xi_2 \quad (3.9)$$

Bearing in mind inequality (3.3) for the normal derivative of the function $e''(\xi)$ and inequality (3.4) for the boundary values of the function $e'(\xi)$, we deduce from Eq. (3.9) that $\kappa_1'' - \kappa_1' > 0$, whence the required inequality (3.8) immediately follows.

Note that since, by construction, the domain ω_1 is contained in the square $|\xi_1| < l$, $|\xi_2| < l$, the estimates

$$\kappa_1 < \ln(l/2), \quad c_{\log}(\omega_1) < l/2$$

follow from consideration of the two-dimensional problem.

The domain ω_1 is the contact area in expanded coordinates (2.6). Therefore, as the logarithmic capacity of the actual contact area ω_ε, obtained by compressing the domain ω_1 by a factor of $1/\varepsilon$, we must put $c_{\log}(\omega_\varepsilon) = \varepsilon c_{\log}(\omega_1)$.

4. THE INTEGRAL EQUATION FOR THE AVERAGE DENSITY OF LINEAR PRESSURES

The solution of boundary-layer problem (2.7)–(2.12) may be represented in the form

$$w(s;\xi) = -\delta_0 - \beta_2 f_1(s) + \beta_1 f_2(s) - \pi^{-1}\gamma(s)e(\xi) \quad (4.1)$$

The harmonic function (2.8) exactly satisfies the boundary conditions (2.8), (2.9) and the periodicity conditions (2.11). By formula (3.1), we obtain

$$w(s;\xi) = w_0(s) - \pi^{-1}\gamma(s)[-\ln\rho + \kappa_1] + O(\rho^{-1}), \quad \rho\to\infty \quad (4.2)$$

Now, equating terms $O(1)$ in expansions (4.2) and (2.12), we arrive at the equation

$$\frac{\gamma(s)}{\pi}\left(\ln\frac{\varepsilon}{2l}-J^0(s)\right)-\frac{1}{\pi}(J\gamma)(s) = w_0(s)-\frac{\gamma(s)}{\pi}\kappa_1$$

whence, in view of the notation (2.10) and (3.2), we deduce

$$\gamma(s)\left(\ln\frac{2l}{\varepsilon c_{\log}(\omega_1)}+J^0(s)\right)+(J\gamma)(s) = -\pi w_0(s) \tag{4.3}$$

The properties of the operator J defined by formula (2.4) were studied in [3, 11]. In particular, it has been shown ([3], Lemma 3) that the operation $J^0\mathbf{1} + J$ has a discrete spectrum which clusters to $-\infty$, its eigenvalues satisfying the asymptotic relation $\lambda_k = -\ln k + O(1)$ as $k \to \infty$. Equation (4.3) is therefore not available for all right-hand sides if the value of the parameter ε is such that $\ln[\varepsilon c_{\log}(\omega_1)/(2l) + \lambda_k = 0$. Accordingly, for an infinitesimal sequence $\{\varepsilon_k\}$ the required density γ cannot generally be determined directly from Eq. (4.3).

Various constructions of an asymptotic solution of the resultant equation (4.3) have been proposed [12, 11, 3] to overcome this difficulty; these constructions also yield constructions of the asymptotic behaviour of the solution of the initial problem (1.2)–(1.4). A modification of the matching procedure proposed in [13] yields a rigorously solvable resultant integral equation for the density $\gamma(s)$.

Following the approach described in [13], we apply a coordinate transformation inverse to (2.6) in Eqs (4.1) and (4.2), and match the outer and inner asymptotic representations (2.1) and (4.1) of the function $\varphi(\mathbf{x})$. The principal terms of the asymptotic expansions of the functions $\upsilon(\gamma; s, r, \varphi)$ (as $r \to 0$) and $w(s; \varepsilon^{-1}(s-s^j), \varepsilon^{-1}n, \varepsilon^{-1}x_3)$ (as $\varepsilon^{-1}r \to \infty$) are identical. In the matching domain, where $r/l = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, the asymptotic relation

$$w(s;\varepsilon^{-1}(s-s^j),\varepsilon^{-1}n,\varepsilon^{-1}x_3)-\upsilon(\gamma;s,r,\varphi) = o(1) \tag{4.4}$$

transforms, in accordance with the asymptotic formula (4.2), to

$$w_0(s)+\frac{\gamma(s)}{\pi}\ln\frac{r}{\varepsilon c_{\log}(\omega_1)}+\frac{1}{2\pi}\int_\Gamma\frac{\gamma(t)dt}{R_r(s,t)} = o(1),\quad \frac{r}{l}\sim\sqrt{\varepsilon},\quad \varepsilon\to 0 \tag{4.5}$$

where

$$R_r(s,t)^2 = R_0(s,t)^2+r^2-2r\cos\varphi\{f_2'(s)[f_1(s)-f_1(t)]-f_1'(s)[f_2(s)-f_2(t)]\}$$

Obviously, substitution of expansion (2.2) into (4.5) again leads to Eq. (4.3). In other words, the left-hand side of Eq. (4.5) turns out to be small in the matching zone, if it is equated to zero beneath the base of the punch at a depth $\sqrt{\varepsilon}l$, that is, at $\varphi = 0$ and $r = \sqrt{\varepsilon}l$. In this way one arrives at the equation

$$\gamma(s)\ln\frac{k}{\sqrt{\varepsilon}c_{\log}(\omega_1)}+\frac{1}{2}\int_\Gamma\frac{\gamma(t)dt}{\sqrt{R_0(s,t)^2+\varepsilon l^2}} = -\pi w_0(s) \tag{4.6}$$

Recalling the notation $\varepsilon = 1/N$ and $P(s) = E[2(1-\nu^2)]^{-1}\gamma(s)$, we can rewrite Eq. (4.6) in the final form

$$2P(s)\ln\frac{\sqrt{N}l}{c_{\log}(\omega_1)}+\int_\Gamma\frac{P(t)dt}{\sqrt{R_0(s,t)^2+(l^2/N)}} = -\frac{\pi E}{1-\nu^2}w_0(s) \tag{4.7}$$

By following, for example, the scheme employed in [13], it is not difficult to prove an existence and uniqueness theorem of the solution of integral equation (4.7) for any sufficiently large values of the parameter N.

5. THE ASYMPTOTIC FORM OF THE REDUCED LOGARITHMIC CAPACITY

It was shown in Section 3 that the reduced logarithmic capacity $c_{\log}(\omega_1)$ is a monotone functional of the domain ω_1. We shall now determine the asymptotic form of the quantity $c_{\log}(\omega_1)$ as the diameter of ω_1 tends to zero.

Let μ denote a small positive parameter, and let us introduce the set

$$\omega_\mu = \{(\xi_1, \xi_2): \mu^{-1}(\xi_1, \xi_2) \in \omega_1\}$$

Let us determine the asymptotic form as $\mu \to 0$ of the solution $e_\mu(\xi)$ of the problem, consisting of Eq. (2.7), the boundary conditions

$$\frac{\partial e_\mu}{\partial \xi_3}(\xi) = 0, \quad \xi_3 = 0, \quad (\xi_1, \xi_2) \notin \overline{\omega_\mu}, \quad |\xi_1| < l$$

$$e_\mu(\xi_1, \xi_2, 0) = 0 \quad (\xi_1, \xi_2) \in \omega_\mu$$

the periodicity conditions (2.11) and the asymptotic condition (3.1).

We shall use the method of matched expansions and introduce expanded coordinates

$$\zeta = \mu^{-1}\xi \tag{5.1}$$

(On changing to the coordinates (5.1) the parameter μ is eliminated from the equation of the boundary of the domain ω_μ.)

Let $Y(\zeta)$ and $\mathbf{c}_1$ denote the capacity potential and harmonic capacity of the set $\{\zeta : \zeta_3 = 0, (\zeta_1, \zeta_2) \in \overline{\omega_1}\}$ (see, e.g. [14])

$$\Delta_\zeta Y(\zeta) = 0, \quad \zeta_3 < 0; \quad Y(\zeta_1, \zeta_2, 0) = 1 \cdot \quad (\zeta_1, \zeta_2) \in \omega_1$$

$$\frac{\partial Y}{\partial \zeta_3}(\zeta) = 0, \quad \zeta_3 = 0 \ \ (\zeta_1, \zeta_2) \notin \overline{\omega_1}$$

We have

$$Y(\zeta) = \mathbf{c}_1|\zeta|^{-1} + O(|\zeta|^{-2}), \quad |\zeta| \to \infty \tag{5.2}$$

and the following integral representation holds

$$\mathbf{c}_1 = \frac{1}{2\pi}\iint_{\omega_1} \frac{\partial Y}{\partial \zeta_3}(\zeta_1, \zeta_2, 0-)d\zeta_1 d\zeta_2 \tag{5.3}$$

Since by formula (3.5) the function $e_\mu(\xi)$ must satisfy the equality

$$1 = \frac{\mu}{2\pi l}\iint_{\omega_1} \frac{\partial e_\mu}{\partial \zeta_3}(\mu^{-1}\zeta_1, \mu^{-1}\zeta_2, 0-)d\zeta_1 d\zeta_2$$

the inner asymptotic representation of $e_\mu(\xi)$, valid in the neighbourhood of the domain ω_μ, is taken to be the function

$$\mathcal{W}(\zeta) = \frac{l}{\mu \mathbf{c}_1}(Y(\zeta) - 1) \tag{5.4}$$

In view of formula (5.2), the following expansion holds as $|\zeta| \to \infty$

$$\mathcal{W}(\zeta) = \frac{l}{\mu \mathbf{c}_1}\left(-1 + \frac{\mathbf{c}_1}{|\zeta|} + O(|\zeta|^{-2})\right) \tag{5.5}$$

As outer asymptotic representation, which is supposed to approximate the unknown function $e_\mu(\xi)$ far from the origin, we take the sum

$$\mathcal{V}(\xi) = \frac{1}{2}\sum_{k=-n_\mu}^{n_\mu} f(k; \xi) + A_\mu; \quad f(k; \xi) = \frac{2l}{\sqrt{(\xi_1 - 2kl)^2 + \xi_2^2 + \xi_3^2}} \tag{5.6}$$

where n_μ is a large natural number, which depends on the parameter μ, and A_μ is a constant. Note that the residual left by the function (5.6) in the second matching condition (2.11) is the smaller the greater the number n_μ.

In order to match the outer asymptotic representation (5.6) with the inner one (5.4), let us determine the asymptotic form of the function $\mathcal{V}(\xi)$ as $|\xi| \to 0$. The function (5.6) admits of the expansion

$$\mathcal{V}(\xi) = \frac{l}{|\xi|} + S(n_\mu) + A_\mu + O(|\xi|^2); \quad S(n_\mu) = \sum_{k=1}^{n_\mu} \frac{l}{k}$$

Changing here to expanded coordinates (5.1), we obtain

$$\mathcal{V}(\mu\zeta) = \frac{l}{\mu|\zeta|} + S(n_\mu) + A_\mu + O(\mu^2|\zeta|^2) \tag{5.7}$$

We now associate the two expansions (5.7) and (5.5) with the same function $e_\mu(\xi)$ in the matching zone, where the number $|\zeta|/l$ turns out to be of the order of $\mu^{-1/2}$ as $\mu \to 0$. Thus, the terms on the right of the expansions (5.7) and (5.5) will be entirely identical if we set

$$A_\mu = -\frac{l}{\mu \mathbf{c}_1} - S(n_\mu) \tag{5.8}$$

Substituting expression (5.8) into formula (5.6), we have (the prime means that $j \neq 0$)

$$\mathcal{V}(\xi) = \frac{l}{|\xi|} - \frac{l}{\mu \mathbf{c}_1} + \frac{1}{2} \sum_{k=-n_\mu}^{n_\mu}{}' \left(f(k;\, \xi) - \frac{1}{|k|} \right) \tag{5.9}$$

We can now let the parameter n_μ in the sum (5.9) tend to infinity, since the resulting series is convergent (by the Cauchy integral test).

Thus, the final candidate for constructing the outer asymptotic representation of the function $e_\mu(\xi)$ is the function (5.9) when $n_\mu = \infty$.

We will now proceed to ascertain the behaviour of this function as $\rho = \sqrt{\xi_2^2 + \xi_3^2} \to \infty$. It is clear that if $|\xi_1| \le l$, then $f(|k|;\, \xi) > f|k| + 1;\, \xi)$. We thus have the estimates

$$\int_1^n f(t;\, \xi)dt + f(n;\, \xi) < \sum_{k=1}^n f(k;\, \xi) < \int_1^n f(t;\, \xi)dt + f(1;\, \xi)$$

Hence it follows that

$$\sum_{k=1}^n f(k;\, \xi) = \int_1^n f(t;\, \xi)dt + R_n(\xi), \quad R_n(\xi) \in (f(n;\, \xi), f(1;\, \xi)) \tag{5.10}$$

On the other hand, the following expansion is well known (see, e.g. [15, Ch. 8, Section 3])

$$S(n) = \sum_{k=1}^n \frac{1}{k} = \int_1^n \frac{dt}{t} + \mathbf{C} + O(n^{-1}), \quad n \to \infty \tag{5.11}$$

where $\mathbf{C} = 0.577\ldots$ is Euler's constant.

Combining relations (5.10) and (5.11) and letting $n \to \infty$, we obtain

$$\sum_{k=1}^n \left(f(k, \xi) - \frac{1}{k} \right) = \int_1^\infty \left(f(t, \xi) - \frac{1}{t} \right) dt - \mathbf{C} + O(f(1, \xi))$$

Thus, the function (5.9) admits of the following asymptotic expansion when $n_\mu = \infty$.

$$\mathcal{V}(\xi) = -\ln\frac{\rho}{4l} - \frac{l}{\mu \mathbf{c}_1} - \mathbf{C} + O(\rho^{-1}), \quad \rho \to \infty \tag{5.12}$$

whence we obtain the following asymptotic formula for the constant κ_μ in expansion (3.1) of the function $e_\mu(\xi)$

$$\kappa_\mu = \ln(4l) - \frac{l}{\mu \mathbf{c}_1} - \mathbf{C} + o(1), \quad \mu \to 0 \tag{5.13}$$

We thus arrive at the final result

$$c_{\log}(\omega_\mu) = 4l \exp\left(-\frac{l}{\mu \mathbf{c}_1} - \mathbf{C}\right)(1 + o(1)), \quad \mu \to 0 \tag{5.14}$$

It can be shown that the remainders in asymptotic formulae (5.13) and (5.14) estimated as $o(1)$ are actually $O(\mu^2)$.

6. FORMULATION OF THE STRUCTURALLY NON-LINEAR CONTACT PROBLEM

Let us imagine that along the contour Γ there are N spherical punches of radius R, linked together by a rigid ring, as defined by the equations

$$x_3 = \Phi_j(x_1, x_2), \quad j = 0, 1, \ldots, N-1 \tag{6.1}$$

$$\Phi_j(x_1, x_2) = (2R)^{-1}[(x_1 - f_1(s_\varepsilon^j))^2 + (x_2 - f_2(s_\varepsilon^j))^2] \tag{6.2}$$

It is natural to assume that the radius R is comparable with the distance between adjacent punches, which is a quantity of the order of l/N for large values of N. We shall assume in addition that each of the punches is displaced by a distance that is small compared with the radius R. The displacement of the punches are determined by the displacement of the ring, which is characterized by the following parameters: δ_0 is the translational displacement of the ring in the opposite direction to that of the vertical axis Ox_3, and β_1 and β_2 are the angles of rotation of the ring about the horizontal coordinate axes Ox_1 and Ox_2. Letting $\varepsilon = 1/N$ denote, as before, a small parameter, we specify the previous assumptions as follows:

$$\delta_0 = \varepsilon^2 \delta_0^*, \quad \beta_i = \varepsilon^2 \beta_i^*, \quad i = 1, 2; \quad R = \varepsilon R^* \tag{6.3}$$

where the quantities δ_0^*, β_1^*, β_2^* and R^* are comparable with the quantity l as $\varepsilon \to 0$.

By the Papkovich–Neuber representation, the contact problem for the above system of punches pressed without friction into the elastic half-space $x_3 \leq 0$ reduces to the problem for the potential $\varphi(\mathbf{x})$ comprising Eqs (1.2) and (1.4) and the boundary conditions of unilateral contact (see [16], and also [17, 18], etc.)

$$\varphi(\mathbf{x}) \leq -\delta_0 - \beta_2 x_1 + \beta_1 x_2 + \Phi_j(x_1, x_2), \quad \frac{\partial \varphi}{\partial x_3}(\mathbf{x}) \leq 0$$

$$[\varphi(\mathbf{x}) + \delta_0 + \beta_2 x_1 - \beta_1 x_2 - \Phi_j(x_1, x_2)]\frac{\partial \varphi}{\partial x_3}(\mathbf{x}) = 0 \tag{6.4}$$

$$x_3 = 0 \quad (x_1, x_2) \in \omega_\varepsilon^j \quad (j = 0, 1, \ldots, N-1)$$

where ω_ε^j is a domain that surely covers the contact area beneath the jth punch. Since there is certainly no contact wherever the punch surface is situated above the level of the unperturbed boundary of the elastic base, we can put

$$\omega_\varepsilon^j = \{(x_1, x_2) \colon \delta_0 + \beta_2 x_1 - \beta_1 x_2 - \Phi_j(x_1, x_2) > 0\}$$

Under these conditions, due to (6.3), the diameter of the domain ω_ε^j turns out to be $O(\varepsilon^{3/2} l)$ and is small compared with the distance between punches.

In order to construct the principal terms of the asymptotic expansion of the solution of the above problem in explicit form (to simplify certain arguments when deriving the unilateral contact problem for the boundary layer), expression (6.2) is replaced by the asymptotically equivalent expression

$$\Phi_j(x_1, x_2) = (2R)^{-1}[(s - s_\varepsilon^j)^2 + n^2] \tag{6.5}$$

where s and n are local coordinates in the neighbourhood of the curve Γ, and s_ε^j is the coordinate of the vertex of the punch, defined by the third formula of (2.6).

7. THE CONSTRUCTION OF AN ASYMPTOTIC FORM OF THE STRUCTURALLY NON-LINEAR CONTACT PROBLEM

The outer asymptotic representation (2.1) is left unchanged, since far from the punches the distinctive features of their bases are levelled out. As before, the density $\gamma(s)$ is assumed to depend on the parameter ε, but this will not enter in the notation.

The problem for the boundary layer comprises the Laplace equation (2.7), the periodicity condition (2.11) and the following boundary condition of unilateral contact, obtained from condition (6.4) by considering expression (6.5)

$$\begin{gathered} w(s;\xi) \le \varepsilon^2 w_0^*(s) + \varepsilon\Phi^*(\xi_1,\xi_2), \quad \frac{\partial w}{\partial \xi_3}(s;\xi) \le 0 \\ [w(s;\xi) - \varepsilon^2 w_0^*(s) - \varepsilon\Phi^*(\xi_1,\xi_2)]\frac{\partial w}{\partial \xi_3}(s;\xi) = 0, \quad \xi_3 = 0, \quad |\xi_1| < l \end{gathered} \tag{7.1}$$

where we have introduced expanded coordinates (2.6) and put

$$w_0^*(s) = -\delta_0^* - \beta_2^* f_1(s) + \beta_1^* f_2(s), \quad \Phi^*(\xi_1,\xi_2) = (2R^*)^{-1}(\xi_1^2 + \xi_2^2) \tag{7.2}$$

These relations are completed by adding the asymptotic condition (2.12) obtained by matching on the basis of (2.2).

The solution of the boundary-layer problem, whose boundary condition (7.1) involves the small parameter ε, will be constructed by using matched expansions, applying the approach described previously in [19] and the methods used in Section 5. Thus, as outer asymptotic representation for the function $w(s;\xi)$, proceeding by analogy with (5.9), we designate the function

$$\mathcal{V}(s;\xi) = -\frac{\gamma(s)}{\pi}\left\{\frac{l}{|\xi|} + \frac{1}{2}\sum_{k=-\infty}^{\infty}{}' \left(f(k;\xi) - \frac{1}{|k|}\right)\right\} + A(\varepsilon; s) \tag{7.3}$$

The constant $A(\varepsilon; s)$ is determined, along with the factor preceding the braces in (7.3), by assuming that the function (7.3) satisfies the asymptotic condition (2.12). Namely, in view of expansion (5.12), which holds for the function (5.9), we obtain the following representation of function (7.3)

$$\mathcal{V}(s;\xi) = -\frac{\gamma(s)}{\pi}\left\{-\ln\frac{\rho}{4l} - \mathbf{C}\right\} + A(\varepsilon; s) + O(\rho^{-1}), \quad \rho \to \infty \tag{7.4}$$

Equating the terms written in the asymptotic formulae (7.4) and (2.12), we find that

$$A(\varepsilon; s) = -\frac{\gamma(s)}{\pi}\left(\ln\frac{1}{2\varepsilon} + \mathbf{C} + J^0(s)\right) - \frac{1}{\pi}(J\gamma)(s) \tag{7.5}$$

On the other hand, as $|\xi| \to 0$ we have

$$\mathcal{V}(s;\xi) = -\frac{\gamma(s)}{\pi}\frac{l}{|\xi|} + A(\varepsilon; s) + O(|\xi|^2) \tag{7.6}$$

To construct the inner asymptotic representation $\mathcal{W}(s;\zeta)$ of the function $w(s;\xi)$ we introduce expanded coordinates

$$\zeta = \varepsilon^{-1/2}\xi \tag{7.7}$$

The value of the exponent of ε in (7.7) is chosen so as to equalize the orders of the terms on the right of the first inequality in boundary condition (7.1).

The function $\mathcal{W}(s;\zeta)$ must be harmonic in the half-space $\zeta_3 < 0$ and satisfy on its boundary the following relation, which follows from boundary condition (7.1) when Eq. (7.7) is taken into account

$$\begin{gathered}\mathcal{W}(s;\zeta) \le \varepsilon^2[w_0^*(s) + \Phi^*(\xi_1,\xi_2)], \quad \frac{\partial\mathcal{W}}{\partial\zeta_3}(s;\zeta) \le 0 \\ (\mathcal{W}(s;\zeta) - \varepsilon^2[w_0^*(s) + \Phi^*(\xi_1,\xi_2)])\frac{\partial\mathcal{W}}{\partial\zeta_3}(s;\zeta) = 0, \quad \zeta_3 = 0\end{gathered} \tag{7.8}$$

The need to match the inner asymptotic representation $\mathcal{W}(s;\zeta)$ with the outer asymptotic representation $\mathcal{V}(s;\zeta)$ of the function $w(s;\zeta)$ by (7.6) dictates the following behaviour of $\mathcal{W}(s;\zeta)$ at infinity

$$\mathcal{W}(s;\zeta) = -\varepsilon^{-1/2}\frac{\gamma(s)}{\pi}\frac{l}{|\zeta|} + A(\varepsilon;s) + O(|\zeta|^{-2}), \quad |\zeta| \to \infty \tag{7.9}$$

Consider the simple layer potential

$$\mathcal{W}^*(s;\zeta) = -\frac{1}{2\pi}\int\int_{\omega_1^*(s)}\frac{\tilde{q}^*(s;\eta_1,\eta_2)d\eta_1 d\eta_2}{\sqrt{(\zeta_1-\eta_1)^2 + (\zeta_2-\eta_2)^2 + \zeta_3^2}} \tag{7.10}$$

with Hertz density (for the equations of Hertz's theory see, e.g. [20]):

$$\tilde{q}^*(s;\eta_1,\eta_2) = \frac{3\tilde{Q}^*(s)}{2\pi a_*^2}\sqrt{a_*^2 - \zeta_1^2 - \zeta_2^2} \tag{7.11}$$

distributed over the circular area $\omega_1^*(s)$ with centre at the origin and radius a_*. Function (7.10) satisfies the following boundary condition

$$\mathcal{W}^*(s;\zeta_1,\zeta_2,0) = -\frac{3\tilde{Q}^*(s)}{16a_*^2}(2a_*^2 - \zeta_1^2 - \zeta_2^2) \quad (\zeta_1,\zeta_2) \in \omega_1^*(s) \tag{7.12}$$

Outside the domain $\omega_1^*(s)$ the normal derivative of the potential (7.10) vanishes. On the other hand, we have the expansion

$$\mathcal{W}^*(s;\zeta) = -\frac{1}{2\pi}\frac{\tilde{Q}^*(s)}{|\zeta|} + O(|\zeta|^{-3}), \quad |\zeta| \to \infty \tag{7.13}$$

Thus, the inner asymptotic representation of the function $\mathcal{W}(s;\zeta)$ may be expressed in the form

$$\mathcal{W}(s;\zeta) = \varepsilon^2\mathcal{W}^*(s;\zeta) + A(\varepsilon;s) \tag{7.14}$$

Assuming that the matching condition (7.9) is satisfied and taking expansion (7.13) into account, we obtain

$$\gamma(s) = \varepsilon^{5/2}(2l)^{-1}\tilde{Q}^*(s) \tag{7.15}$$

Next, by (7.12), function (7.14) is equal to the following expression in the contact area $\omega_1^*(s)$.

$$\mathcal{W}(s;\zeta_1,\zeta_2,0) = A(\varepsilon;s) - \varepsilon^2\frac{3\tilde{Q}^*(s)}{8a_*} + \varepsilon^2\frac{3\tilde{Q}^*(s)}{16a_*^3}(\zeta_1^2 + \zeta_2^2) \tag{7.16}$$

On the other hand, by boundary condition (7.8) and the notation (7.2), the following equality must hold in the contact area

$$\mathcal{W}(s;\zeta_1,\zeta_2,0) = \varepsilon^2[w_0^*(s)+(2R^*)^{-1}(\zeta_1^2+\zeta_2^2)] \quad (7.17)$$

Equating expressions (7.16) and (7.17), we obtain relations

$$A(\varepsilon;s)-\varepsilon^2\frac{3\tilde{Q}^*(s)}{8a_*} = \varepsilon^2 w_0^*(s), \quad \frac{1}{2R^*} = \frac{3\tilde{Q}^*(s)}{16a_*^3} \quad (7.18)$$

The radius of the contact area is found from the second equation of (7.18)

$$a_* = \left(\frac{3}{8}\tilde{Q}^*(s)R^*\right)^{1/3} \quad (7.19)$$

Substituting (7.19) into the first equation (7.18), we obtain

$$A(\varepsilon;s)-\varepsilon^2\left(\frac{9\tilde{Q}^*(s)^2}{64R^*}\right)^{1/3} = \varepsilon^2 w_0^*(s) \quad (7.20)$$

Thus, all the arbitrariness admitted in the asymptotic constructions has been eliminated. To determine the functions $\gamma(s)$, $\tilde{Q}^*(s)$ and $A(\varepsilon;s)$ we now have the system of equations (7.5), (7.15) and (7.20).

8. THE EQUATION FOR THE AVERAGE DENSITY OF LINEAR PRESSURES IN THE STRUCTURALLY NON-LINEAR PROBLEM

Using Eq. (7.15), let us express the quantity $\tilde{Q}^*(s)$ in terms of $\gamma(s)$, substitute the result into Eqs (7.5) and (7.20), and then eliminate the quantity $A(\varepsilon;s)$ from the system thus obtained. The yields the equation

$$m\gamma(s)^{2/3}+\gamma(s)(|\ln\varepsilon|-\ln 2+\mathbf{C}+J^0(s))+(J\gamma)(s) = -\pi w^0(s) \quad (8.1)$$

where we have introduced the notation

$$m = \pi(9l^2)^{1/3}(16R)^{-1/3}$$

The function $w^0(s)$ is defined by formula (2.10).

The question of the solvability Eq. (8.1) and the construction of an asymptotic expansion of its solution as $\varepsilon\to 0$ (in view of the properties of the operator J) remains open.

To obtain a rigorously solvable resultant equation for the function $\gamma(s)$, we shall use a modified matching procedure [13]. Consider the asymptotic expansion (7.4) of the function $\mathcal{V}(\varepsilon;\xi)$ at infinity. We apply a coordinate transformation inverse to (2.6) in formula (7.4), reverting from expanded coordinates to real coordinates, and then, using the resulting relation, express the matching condition (4.4) in the form

$$\frac{\gamma(s)}{\pi}\left(\ln\frac{r}{\varepsilon 4l}+\mathbf{C}\right)+A(\varepsilon;s)+\frac{1}{2\pi}\int_\Gamma\frac{\gamma(t)dt}{R_r(s,t)} = o(1), \quad \frac{r}{l} = O(\sqrt{\varepsilon}), \quad \varepsilon\to 0 \quad (8.2)$$

Now, equating the left-hand side of (8.2) to zero beneath the base of the punch for $r=\sqrt{\varepsilon}l$, we obtain an equation linking the quantities $A(\varepsilon;s)$ and $\gamma(s)$, instead of Eq. (7.5). Thus, instead of (8.1) we obtain the following equation for the density $\gamma(s) = 2(1-\nu^2)E^{-1}P(s)$ of linear pressures

$$m\gamma(s)^{2/3}+\gamma(s)\left(\ln\frac{1}{4\sqrt{\varepsilon}}+\mathbf{C}\right)+\frac{1}{2\pi}\int_\Gamma\frac{\gamma(t)dt}{\sqrt{R_0(s,t)^2+\varepsilon l^2}} = -\pi w_0(s) \quad (8.3)$$

That this equation is solvable may be established by reducing it to a Hammerstein integral equation (see, e.g. [21]).

9. CONCLUSION

The function $P(s)$ defines the contact pressure per unit length of the arc of Γ. The contact pressure developed under the base of the punch ω_ε^j is determined in expanded coordinates from the solution of the boundary-layer problem via formulae (2.13) and (7.11).

The main results of this paper are Eqs (4.7) and (8.3) for the average density of contact pressures, inequality (3.8) and asymptotic formula (5.14) for the reduced logarithmic capacity of the contact area. The essential point when constructing the asymptotic forms for the structurally non-linear contact problem for the boundary layer was the construction of the outer asymptotic representation in the form (7.4). By applying a procedure worked out in [19, 22], Eq. (7.18) may be improved, which of course involves the need to modify the resultant equation (8.3).

Note that Eqs (4.7) and (8.3) remain valid when Γ is a simple (non-self-intersecting) smooth open curve. However, a three-dimensional boundary layer is formed in the neighbourhood of the ends of the arc Γ. This topic has not been investigated for problems of the class under consideration.

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